

Brauer algebras of type $I_2^n (n \geq 5)$

Shoumin Liu

Abstract

We will present an algebra related to the Coxeter group of type I_2^n which can be taken as a twisted subalgebra in Brauer algebra of type A_{n-1} . Also we will describe some properties of this algebra.

1 Introduction

From studying the invariant theory for orthogonal groups, Brauer discovered Brauer algebras of type A in [2]; Cohen, Frenk and Wales extended it to the definition of simply laced type in [4]. Mühlherr described how to get Coxeter group of type I_2^n by twisting Coxeter group of type A_{2n-1} in [7]. Here we will apply the similar approach as Mühlherr on $\text{Br}(A_{n-1})$, to get $\text{Br}(I_2^n)$. We give the definition of $\text{Br}(I_2^n)$ as follows.

Definition 1.1. *The Brauer algebra of type I_2^{2m} for $m \in \mathbb{N}_{>2}$, denoted by $\text{Br}(I_2^{2m})$, is a unital associative $\mathbb{Z}[\delta^{\pm 1}]$ -algebra generated by r_0, r_1, e_0 and e_1 subject to the following relations and a set $\Theta \subset \mathbb{N}$ consisting of $\kappa_i, \eta_j, \xi_j, \theta_j$, where $i = 0, 1, j = 1, \dots, [m/2]$. Symbols $[r_0 r_1 \cdots]_t$ and $[r_1 r_0 \cdots]_t$ stand for words of length t with r_0 and r_1 iterated.*

$$r_i^2 = 1, \quad (1.1)$$

$$r_i e_i = e_i r_i = e_i, \quad (1.2)$$

$$e_i^2 = \delta^{\kappa_i} e_i, \quad (1.3)$$

$$e_1 e_0 e_1 = \delta e_1, \quad (1.4)$$

$$e_0 [r_1 r_0 \cdots]_{2m-1} = [r_1 r_0 \cdots]_{2m-1} e_0, \quad (1.5)$$

$$e_1 [r_0 r_1 \cdots]_{2m-1} = e_1, \quad (1.6)$$

$$[r_0 r_1 \cdots]_{2m-1} e_1 = e_1, \quad (1.7)$$

$$e_0 [r_1 r_0 \cdots]_{2k} e_1 = \delta^{\theta_k} e_0 e_1, \quad 0 \leq k \leq [m/2] \quad (1.8)$$

$$e_1 [r_0 r_1 \cdots]_{2k} e_0 = \delta^{\theta_k} e_1 e_0, \quad 0 \leq k \leq [m/2] \quad (1.9)$$

$$e_1 [r_0 r_1 \cdots]_{2k-1} e_1 = \delta^{\eta_k} e_1, \quad 0 \leq k \leq [m/2] \quad (1.10)$$

$$[r_1 r_0 \cdots]_{2m} = [r_0 r_1 \cdots]_{2m} \quad (1.11)$$

and when $2k \leq m$, let $l = \text{lcm}(k, m)$,

$$e_0[r_1 r_0 \cdots]_{2k-1} e_0 = \delta^{\xi_k} [r_1 r_0 \cdots]_{2m-1} e_0, \quad \frac{l}{m}, \frac{l}{k} \text{ odd}, \quad (1.12)$$

$$e_0[r_1 r_0 \cdots]_{2k-1} e_0 = \delta^{\xi_k} e_0, \quad \frac{l}{m} \text{ even}, \quad (1.13)$$

$$e_0[r_1 r_0 \cdots]_{2k-1} e_0 = \delta^{\xi_k} e_0 e_1 e_0, \quad \frac{l}{m} \text{ odd}, \frac{l}{k} \text{ even}. \quad (1.14)$$

The submonoid of the multiplicative monoid of $\text{Br}(\mathbb{I}_2^m)$ generated by δ , $\{r_i\}_{i=0}^1$ and $\{e_i\}_{i=0}^1$ is denoted by $\text{BrM}(\mathbb{I}_2^m)$. This is the monoid of monomials in $\text{Br}(\mathbb{I}_2^m)$.

Definition 1.2. *The Brauer algebra of type \mathbb{I}_2^{m-1} for $m \in \mathbb{N}_{>2}$, denoted by $\text{Br}(\mathbb{I}_2^{2m-1})$, is a unital associative $\mathbb{Z}[\delta^{\pm 1}]$ -algebra generated by r_0, r_1, e_0 and e_1 subject to the following relations and a set $\Theta \subset \mathbb{N}$ consisting of κ_i, ξ_j with $j = 1, \dots, m, i = 0, 1, \kappa_0 = \kappa_1$.*

$$r_i^2 = 1, \quad (1.15)$$

$$r_i e_i = e_i r_i = e_i, \quad (1.16)$$

$$e_i^2 = \delta^{\kappa_i} e_i, \quad (1.17)$$

$$[r_0 r_1 \cdots]_{2m-2} e_0 = e_1 [r_0 r_1 \cdots]_{2m-2}, \quad (1.18)$$

$$e_0[r_1 r_0 \cdots]_{2k-1} e_0 = \delta^{\xi_k} e_0, \quad 0 < k < m \quad (1.19)$$

$$[r_1 r_0 \cdots]_{2m-1} = [r_0 r_1 \cdots]_{2m-1}. \quad (1.20)$$

The submonoid of the multiplicative monoid of $\text{Br}(\mathbb{I}_2^{2m-1})$ generated by $\delta, r_0, r_1, e_0, e_1$ is denoted by $\text{BrM}(\mathbb{I}_2^{2m-1})$. This is the monoid of monomials in $\text{Br}(\mathbb{I}_2^{2m-1})$.

It is well known that Coxeter group of type \mathbb{I}_2^n , denoted by $W(\mathbb{I}_2^n)$ can be gotten as a subgroup from Coxeter group of type A_{n-1} denoted by $W(A_{n-1})$ for an special partition on the Coxeter diagram of type A_{n-1} . The following is the partition to get \mathbb{I}_2^6 from A_5 . The main theorem of this paper can be stated as the following.

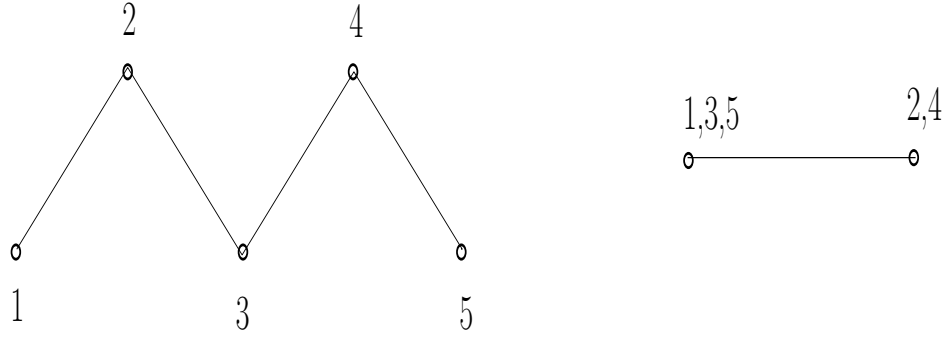
Theorem 1.3. *For $n > 4$, there is an algebra isomorphism*

$$\phi : \text{Br}(\mathbb{I}_2^n) \longrightarrow \text{Br}(A_{n-1})$$

determined by $\phi(r_0) = \prod_{i \text{ even}}^{0 < i < n} R_i$, $\phi(r_1) = \prod_{i \text{ odd}}^{0 < i < n} R_i$, $\phi(e_0) = \prod_{i \text{ even}}^{0 < i < n} E_i$ and $\phi(e_1) = \prod_{i \text{ odd}}^{0 < i < n} R_i$ when each parameter in Θ takes special value in \mathbb{N} to make ϕ an algebra homomorphism. Furthermore,

$$\text{rank}_{\mathbb{Z}[\delta^{\pm 1}]} \text{Br}(\mathbb{I}_2^n) = \begin{cases} 2n + n^2, & \text{if } n \text{ is odd,} \\ 2n + \frac{3}{2}n^2, & \text{if } n \text{ is even.} \end{cases}$$

This paper is included as Chapter 5 in the author's PhD thesis [6].



2 An interesting elementary problem

Suppose that $k, m \in \mathbb{N}$ are such that $1 < 2k \leq m$. There is a box in the x, y plane \mathbb{R}^2 fixed by four lines $x = 1$, $x = 2m$, $y = 2k - \frac{1}{2}$, and $y = -\frac{1}{2}$. Imagine you have a particle, which starts to move from $(1, 2k - 1)$ with slope -1 ; when it touches the bottom (the top), it will be reflected with the bottom (the top) as a mirror; but when it touches the right (left) wall, it first goes down (up) 1 unit vertically, if it comes at the wall from the top (bottom), and continues its path with the wall as the mirror; it stops if it reaches the points $(1, 0)$, $(2m, 0)$, $(1, 2k - 1)$, or $(2m, 2k - 1)$. For different values of k, m , the problem is to decide at which point the particle stops. One example is Figure 1, when $m = 5, k = 2$.

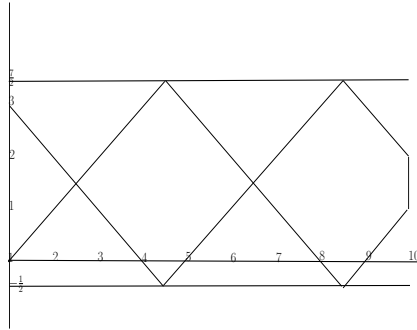


Figure 1: case for $m = 5, k = 2$

To solve the problem, we unfold its path by "penetrating" the walls, which means that when the particle touches the right wall for the first time, we change the vertical step into one move of slope -1 when coming from the top (slope 1 when coming from the bottom) with Euclid length $\sqrt{2}$, or, in other words, we do not change its moving at the wall, and we see that the path of the particle in the region between $x = 2m + 1$ and $x = 4m$ is just the reverse of the path of the particle when it goes from the right to the left for

the first time. The algorithm can be similarly extended at the left wall to make the unfolded path look like the graph of a function of a single variable. It can be verified that when it passes the point with the x -coordinate being a multiple of $2m$, the movement stops. It can be seen that before it stops, the path in $[2tm + 1, 2(t + 1)m]$ is just a copy of a particle path in the above box of :

- (i) the $\frac{t+2}{2}$ th path from the left wall to the right wall if t is even,
- (ii) or $\frac{t+1}{2}$ th path from the right wall to the the left wall if t is odd.

Therefore the above trick is just that we draw the picture on folded paper, then we unfold this and see a simple picture in which the original problem can be tackled. Here is an example for $m = 5$, $k = 2$ in Figure 2 being the unfolded case of Figure 1.

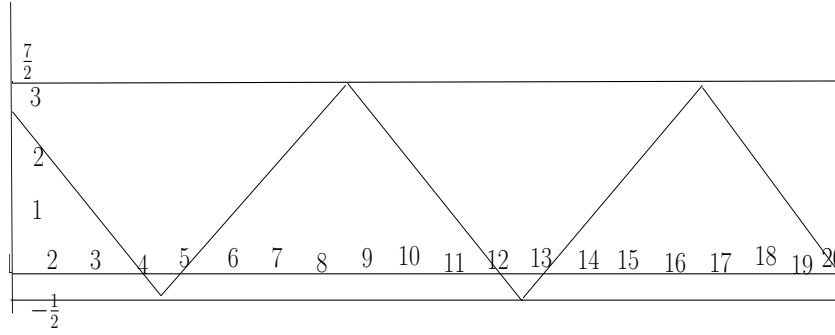


Figure 2: the unfolded path for $m = 5$, $k = 2$

Lemma 2.1. *Let $l = \text{lcm}(k, m)$. The particle stops in the unfolded path when it moves $2l - 1$ for its x -coordinate. Furthermore*

- (i) *when $\frac{l}{m}$ is even, the particle stops at $(1, 0)$;*
- (ii) *when $\frac{l}{m}$ and $\frac{l}{k}$ are odd, the particle stops at $(2m, 0)$;*
- (iii) *when $\frac{l}{m}$ is odd and $\frac{l}{k}$ is even, the particle stops at $(2m, 2k - 1)$.*

Proof. By prolonging the path at the beginning and the ending, respectively, by half unit for x -coordinate to complete a period, we can consider the particle starting from the top and stopping at the top or the bottom. By observing the unfolded path of the particle, each time it goes from the top ceiling to the bottom ground or from the bottom to the top, the x -coordinate is increased by $2k$, so the first conclusion follows naturally. The other two conclusions hold easily by basic number theory knowledge about congruence. Furthermore, elementary number theory tells us that $\frac{l}{m}$ and $\frac{l}{k}$ can not both be even, which implies that the particle never stops at $(1, 2k - 1)$. \square

3 The map ϕ inducing a homomorphism

In order to avoid confusion with the above generators, the symbols of [4] have been capitalized.

Definition 3.1. Let Q be a graph. The Brauer monoid $\text{BrM}(Q)$ is the monoid generated by the symbols R_i and E_i , for each node i of Q and δ, δ^{-1} subject to the following relation, where \sim denotes adjacency between nodes of Q .

$$\delta\delta^{-1} = 1 \quad (3.1)$$

$$R_i^2 = 1 \quad (3.2)$$

$$R_i E_i = E_i R_i = E_i \quad (3.3)$$

$$E_i^2 = \delta E_i \quad (3.4)$$

$$R_i R_j = R_j R_i, \text{ for } i \approx j \quad (3.5)$$

$$E_i R_j = R_j E_i, \text{ for } i \approx j \quad (3.6)$$

$$E_i E_j = E_j E_i, \text{ for } i \approx j \quad (3.7)$$

$$R_i R_j R_i = R_j R_i R_j, \text{ for } i \sim j \quad (3.8)$$

$$R_j R_i E_j = E_i E_j, \text{ for } i \sim j \quad (3.9)$$

$$R_i E_j R_i = R_j E_i R_j, \text{ for } i \sim j \quad (3.10)$$

The Brauer algebra $\text{Br}(Q)$ is the the free \mathbb{Z} -algebra for Brauer monoid $\text{BrM}(Q)$.

In [2], Brauer gives a diagram description for a basis of Brauer monoid of type A_t , which is a monoid consisting of diagrams with $2t + 2$ dots and $t + 1$ strands, where each dot is connected by a unique strand to another dot.

Here we suppose the $2t+2$ dots have coordinates $(i, 0)$ and $(i, 1)$ in \mathbb{R}^2 with $1 \leq i \leq t + 1$. The multiplication of two diagrams is given by concatenation, where any closed loops formed are replaced by a factor of δ . Henceforth, we identify $\text{Br}(A_t)$ with its diagrammatic version. It is a free algebra over $\mathbb{Z}[\delta^{\pm 1}]$ of rank $(t + 1)!!$, the product of the first $t + 1$ positive odd integers.

we revise the root system of the Coxeter group of type A_t , focussing on special collections of mutually orthogonal positive roots called admissible sets. Also, the notion of height for elements of the Brauer algebra $\text{Br}(A_t)$ is introduced and discussed.

Definition 3.2. Let $t \geq 1$. The root system of the Coxeter group $W(A_t)$ of type A_t is denoted by Φ . It is realized as $\Phi := \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq t + 1, i \neq j\}$ in the Euclidean space \mathbb{R}^{t+1} , where ϵ_i is the i^{th} standard basis vector. Put $\alpha_i := \epsilon_i - \epsilon_{i+1}$. Then $\{\alpha_i\}_{i=1}^t$ is called the set of simple roots of Φ . Denote by Φ^+ the set of positive roots in Φ with respect to these simple roots; that is, $\Phi^+ := \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq t + 1\}$.

We have seen that, up to powers of δ , the monomials of $\text{Br}(A_t)$ correspond to Brauer diagrams. In order to work with the tops and bottoms of Brauer diagrams, we introduce the following notion.

Definition 3.3. Let \mathcal{A} denote the collection of all subsets of Φ^+ containing mutually orthogonal positive roots. Members of \mathcal{A} are called *admissible sets*.

An admissible set B corresponds to a Brauer diagram top in the following way: for each $\beta \in B$, where $\beta = \epsilon_i - \epsilon_j$ for some $i, j \in \{1, \dots, t+1\}$, draw a horizontal strand in the corresponding Brauer diagram top from the dot $(i, 1)$ to the dot $(j, 1)$. All horizontal strands on the top are obtained this way, so there are precisely $|B|$ horizontal strands.

For any $\beta \in \Phi^+$ and $i \in \{1, \dots, t\}$, there exists a $w \in W(A_t)$ such that $\beta = w\alpha_i$. Then $E_\beta := wE_iw^{-1}$ is well defined (see [4]). If $\beta, \gamma \in \Phi^+$ are mutually orthogonal, then E_β and E_γ commute (see Lemma 4.3 of [4]). Hence, for $B \in \mathcal{A}$, we can define

$$\hat{E}_B := \delta^{-|B|} \prod_{\beta \in B} E_\beta.$$

This is an idempotent element of the Brauer monoid.

In [4], an action of the Brauer monoid $\text{BrM}(A_t)$ on the collection \mathcal{A} is defined as follows. The generators $\{R_i\}_{i=1}^t$ act by the natural action of Coxeter group elements on its root sets, where negative roots are negated so as to obtain positive roots, and the action of $\{E_i\}_{i=1}^t$ is defined below.

$$E_i B := \begin{cases} B & \text{if } \alpha_i \in B, \\ B \cup \{\alpha_i\} & \text{if } \alpha_i \perp B, \\ R_\beta R_i B & \text{if } \beta \in B - \alpha_i^\perp \end{cases} \quad (3.11)$$

Alternatively, this action can be described as follows: complete the top corresponding to B into a Brauer diagram b , without increasing the number of horizontal strands at the top. Now aB is the top of the Brauer diagram ab . We will make use of this action in order to provide a normal form for elements of $\text{BrM}(A_t)$.

Remark 3.4. There is an anti-involution on $\text{Br}(A_t)$ determined by

$$\gamma_1 \cdots \gamma_t \mapsto \gamma_t \cdots \gamma_1$$

on products of generators of $\text{Br}(A_n)$. We denote it by $x \mapsto x^{\text{op}}$.

Lemma 3.5. *The map ϕ defined on the generators in Theorem 1.3 induces a $\mathbb{Z}[\delta^{\pm 1}]$ -algebra homomorphism.*

Proof. We deal first with n is odd. In Definition 1.2, the nontrivial relations to be verified are just (1.18), (1.19), and (1.20). By [7], the relation (1.20) holds for the inspection of the images of generators. The relation (1.19) follows by observing the diagrams in $\text{Br}(A_{2m-2})$ because the top and the bottom are fixed, both of which have just one point having a vertical strand. By diagram we see that the action of $\phi([r_1 r_0 r_1 \dots]_s)$ in $W(A_{2m-2})$ (which we identify with Sym_{2m-1}) on a number $1 \leq a \leq 2m-1$, for $s \leq 2m-2$, is given by

$$\phi([r_1 r_0 r_1 \dots]_s)(a) = \begin{cases} a + s, & a \text{ odd}, a + s \leq 2m - 1 \\ a - s, & a \text{ even}, a - s \geq 1 \\ 4m - 1 - a - s, & a \text{ odd}, a + s > 2m - 1 \\ 1 + s - a, & a \text{ even}, a - s < 1 \end{cases} \quad (3.12)$$

Then for $t > 0$,

$$\phi([r_1 r_0 r_1 \dots]_{2m-2})(2t) = 2m - 1 - 2t,$$

$$\phi([r_1 r_0 r_1 \dots]_{2m-2})(2t + 1) = 2m - 1 - 2t + 1.$$

Therefore

$$\begin{aligned} \phi([r_1 r_0 r_1 \dots]_{2m-2}^{-1})\alpha_{2t} &= \alpha_{2m-1-2t}, \\ \phi([r_1 r_0 r_1 \dots]_{2m-2}^{-1})\{\alpha_{2t}\}_{t=1}^{m-1} &= \{\alpha_{2t-1}\}_{t=1}^{m-1}, \end{aligned}$$

which implies that $\phi([r_1 r_0 r_1 \dots]_{2m-2}^{-1})\phi(e_0)\phi([r_1 r_0 r_1 \dots]_{2m-2}) = \phi(e_1)$ and (1.18) holds for the generator images under ϕ .

Now consider ϕ when $n = 2m > 5$ even. The fact that (1.1)–(1.4), and (1.8)–(1.11) still hold for the generator images under ϕ can be proved easily by Brauer diagrams. As above we see that

$$\phi([r_1 r_0 r_1 \dots]_s)(a) = \begin{cases} a + s, & a \text{ odd}, a + s \leq 2m - 1 \\ a - s, & a \text{ even}, a - s \geq 1 \\ 4m + 1 - a - s, & a \text{ odd}, a + s > 2m - 1 \\ 1 + s - a, & a \text{ even}, a - s < 1, \end{cases} \quad (3.13)$$

$$\phi([r_0 r_1 r_0 \dots]_s)(a) = \begin{cases} a + s, & a \text{ even}, a + s \leq 2m - 1 \\ a - s, & a \text{ odd}, a - s \geq 1 \\ 4m + 1 - a - s, & a \text{ even}, a + s > 2m - 1 \\ 1 + s - a, & a \text{ odd}, a - s < 1. \end{cases} \quad (3.14)$$

By diagram inspection, we see that

$$\phi([r_1 r_0 r_1 \dots]_{2m-1}^{-1})(\alpha_{2t}) = \alpha_{2m-2t},$$

$$\phi([r_1 r_0 r_1 \dots]_{2m-1}^{-1})(\{\alpha_{2t}\}_{t=1}^{m-1}) = \{\alpha_{2t}\}_{t=1}^{m-1},$$

hence $\phi([r_1 r_0 r_1 \dots]_{2m-1})\phi(e_0)\phi([r_1 r_0 r_1 \dots]_{2m-1}) = \phi(e_0)$; therefore relation (1.5) holds for the images of the generators under ϕ . On the other hand,

$$\phi([r_0 r_1 r_0 \dots]_{2m-1}^{-1})(\alpha_{2t-1}) = \alpha_{2m-2t+1},$$

$$\phi([r_0 r_1 r_0 \dots]_{2m-1}^{-1})(\{\alpha_{2t-1}\}_{t=1}^m) = \{\alpha_{2t}\}_{t=1}^{m-1}.$$

Just observing Brauer diagrams, (1.6) and (1.7) hold for the images of the generators under ϕ .

As for (1.12)–(1.14), consider the top and the bottom of the diagram of the left under ϕ ; both the top and the bottom have horizontal the same strands among those points $\{(i, 0)\}_{i=2}^{2m-1} \cup \{(i, 1)\}_{i=2}^{2m-1}$ as $\phi(e_0)$. Except those $2m-2$ strands in the top and in the bottom fixed for $\phi(e_0)$, there are still two strands of the left side under ϕ unknown. Those two strands are between the remaining four points, $(0, 1)$, $(0, 2m)$, $(1, 1)$ and $(1, 2m)$. If we find another end of the strand from $(1, 1)$, the other strand is fixed as a consequence. The strands starting from $(1, 1)$ in the images of under ϕ the right hand sides of (1.12)–(1.14) are ended at $(0, 2m)$, $(0, 1)$ and $(1, 2m)$, respectively. By observation, we can transform this equality problem to the elementary problem solved at the beginning of this section in the following way. Consider the paths of a particle starting from $(1, 1)$ in the diagram of the left hand sides of the images under ϕ of (1.12)–(1.14) with the $m-1$ horizontal strands at the top and the $m-1$ horizontal strands at the bottom removed and transform the horizontal strands as in Figure 3. We give an example for this in Figure 4 for $\phi(e_0)\phi(r_1)\phi(r_0)\phi(r_1)\phi(e_0)$ in $\text{Br}(A_7)$. By observation, Lemma 2.1 can be applied here, and gives that the three equalities hold under ϕ acting on both sides. \square

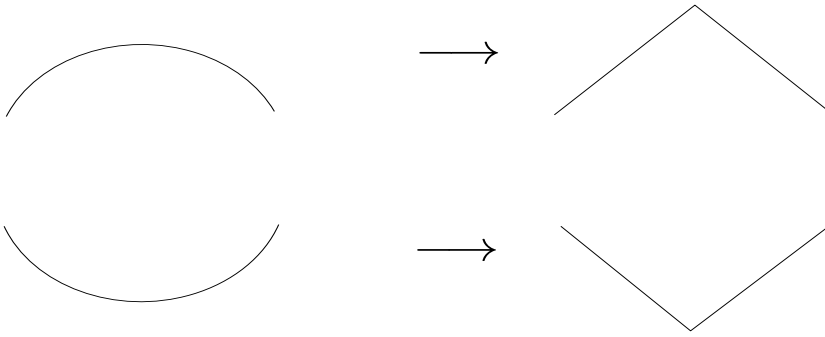


Figure 3: transformation of horizontal strands

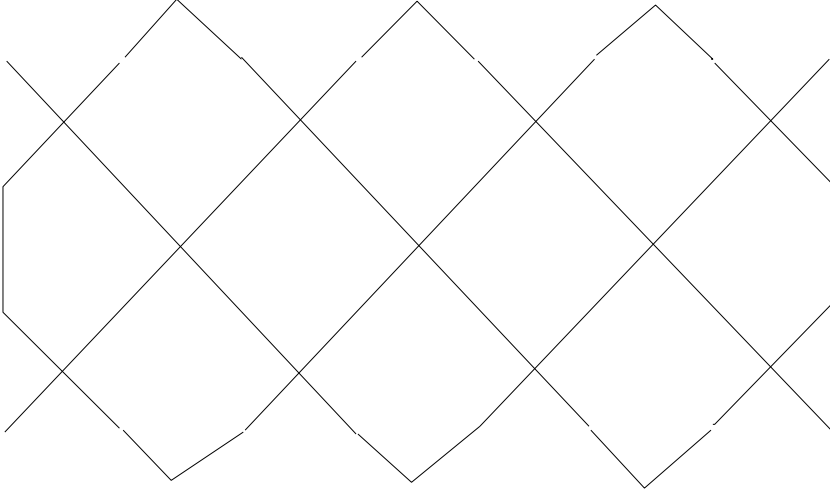


Figure 4: $\phi(e_0)\phi(r_1)\phi(r_0)\phi(r_1)\phi(e_0)$ for I_2^8

4 Normal forms for $\text{BrM}(I_2^n)$

Lemma 4.1. *The submonoid generated by r_0 and r_1 in $\text{BrM}(I_2^n)$ is isomorphic to $W(I_2^n)$.*

Proof. The lemma follows from the natural homomorphisms chain below.

$$\mathbb{Z}[\delta^{\pm 1}](W(I_2^n)) \rightarrow \text{Br}(I_2^n) \rightarrow \text{Br}(I_2^n)/(e_0, e_1) \rightarrow \mathbb{Z}[\delta^{\pm 1}](W(I_2^n)).$$

The composition is the identity and so the lemma follows. \square

From now on, we do not distinguish the $W(I_2^n)$ in $\text{BrM}(I_2^n)$ and its image under ψ .

Analogue to Remark 3.4, we can also define an anti-involution $\text{Br}(I_2^n)$ denoted by $x \mapsto x^{\text{op}}$.

Proposition 4.2. *The natural anti-involution above induces an automorphism of the $\mathbb{Z}[\delta^{\pm 1}]$ -algebra $\text{Br}(I_2^n)$.*

Proof. It suffices to check the defining relations given in Definition 1.1 and Definition 1.2 still hold under the anti-involution. An easy inspection shows that all relations involved in the definition are invariant under op , except for (1.6), (1.7), (1.8), (1.9), and (1.12). The relation obtained by applying the anti-involution to (1.12) holds due to (1.5). The equalities (1.6) and (1.8) are the op -duals of (1.7) and (1.9), respectively. Hence our claim holds. \square

Proposition 4.3. *Suppose that D_0^{2m-1} is the left coset of the subgroup generated by r_0 in $W(I_2^{2m-1})$. Up to some powers of δ , each element in $\text{BrM}(I_2^{2m-1})$ can be written as an element in $W(I_2^{2m-1})$ or ue_0v , where $u \in D_0^{2m-1}$ and $v \in (D_0^{2m-1})^{\text{op}}$.*

Proof. By (1.18), it follows that e_1 is conjugate to e_0 under $W(I_2^{2m-1})$; hence we only need to prove that $D_0^{2m-1}e_0(D_0^{2m-1})^{\text{op}}$ is closed under multiplication by the generators e_0, r_0 and r_1 up to some power of δ . By Proposition 4.2, and invariance of the set $D_0^{2m-1}e_0(D_0^{2m-1})^{\text{op}}$ under the natural involution, it suffices to prove it is closed under left multiplication. For r_0 and r_1 , we just apply (1.16). For e_0 , it follows from (1.19). \square

Let Ψ be the root system of I_2^n , and Ψ^+ be the positive roots with respect to β_0, β_1 which are roots corresponding to r_0, r_1 , respectively. We consider the natural action of $W(I_2^n)$ on Ψ^+ by negating the negative roots once these appear in this action.

Lemma 4.4. *Let N_0, N_1 be stabilizers of β_0 and β_1 in $W(I_2^{2m})$, respectively. Then for any element $a \in N_i$, we have that $ae_i a^{-1} = e_i$, for $i = 0, 1$.*

Proof. When n is odd, $N_i = \langle r_i \rangle$ for $i = 0, 1$. Hence the lemma holds because

$$r_i e_i r_i \stackrel{(1.16)}{=} e_i.$$

When $n = 2m$ even, we have that $N_0 = \langle r_0, [r_1 r_0 \cdots]_{2m-1} \rangle$ and $N_1 = \langle r_1, [r_0 r_1 \cdots]_{2m-1} \rangle$, hence the lemma holds thanks to the following equalities.

$$\begin{aligned} r_i e_i r_i &\stackrel{(1.2)}{=} e_i, \\ [r_1 r_0 \cdots]_{2m-1} e_0 [r_1 r_0 \cdots]_{2m-1} &\stackrel{(1.5)}{=} e_0, \\ [r_0 r_1 \cdots]_{2m-1} e_1 [r_0 r_1 \cdots]_{2m-1} &\stackrel{(1.6)+(1.7)}{=} e_1. \end{aligned}$$

\square

Consider a positive root β and a node i of type I_2^n . If there exists $w \in W$ such that $w\beta_i = \beta$, then we can define the element e_β in $\text{BrM}(I_2^n)$ by

$$e_\beta = w e_i w^{-1}.$$

The above lemma implies that e_β is well defined.

Lemma 4.5. *Let D_i^{2m} be a left coset representatives for N_i in $W(I_2^{2m})$ for $i = 0, 1$, and $K_0 = \langle [r_1 r_0 \cdots]_{2m-1} \rangle \subset N_0$, $K_1 = \langle 1 \rangle \subset N_1$. Then for any $r \in W(G_2)$, there exist $a \in D_i$ and $b \in K_i$, such that*

$$r e_i = a e_i b.$$

Proof. It is a direct result for (1.2), (1.5) and (1.7). \square

Proposition 4.6. *Up to some power of δ , each element in $\text{BrM}(I_2^{2m})$ can be written as*

- (i) $a \in W(I_2^{2m})$,
- (ii) $ue_i v w$, $u \in D_i^{2m}$, $v \in K_i$, $w \in (D_i^{2m})^{\text{op}}$ for $i = 0, 1$,
- (iii) $u' e_0 e_1 w'$, $u' \in D_0^{2m}$, $w' \in (D_1^{2m})^{\text{op}}$,
- (iv) $u'' e_1 e_0 w''$, $u'' \in D_1^{2m}$, $w'' \in (D_0^{2m})^{\text{op}}$,
- (v) $u''' e_0 e_1 e_0 w'''$, $u''' \in D_0^{2m}$, $w''' \in (D_0^{2m})^{\text{op}}$.

Proof. Let us first prove the claim that the monomial $e_0 r e_1$ can be written as $e_0 e_1$ for any $r \in W(I_2^{2m})$ up to some power of δ . In view of (1.2), we only need consider the elements that can be written as $[r_1 r_0 \cdots]_{2k}$ with $2k \leq 2m$. Also thanks to (1.8), we can restrict ourselves to $m < 2k \leq 2m$, follows from the below.

$$\begin{aligned}
e_0 [r_1 r_0 \cdots]_{2k} e_1 &\stackrel{(1.1)}{=} e_0 [r_0 r_1 \cdots]_{2m-2k-1} [r_0 r_1 \cdots]_{2m-1} e_1 \\
&\stackrel{(1.7)}{=} e_0 r_0 [r_1 r_0 \cdots]_{2m-2k-2} e_1 \\
&\stackrel{(1.2)}{=} e_0 [r_1 r_0 \cdots]_{2m-2k-2} e_1 \\
&\stackrel{(1.8)}{=} \delta^{\theta_{m-k-1}} e_0 e_1.
\end{aligned}$$

It follows that for any $r \in W(I_2^{2m})$, the monomial $e_i r e_j$ can be written as one of e_0 , $[r_1 r_0 \cdots]_{2m-1} e_0$, e_1 , $e_0 e_1$, $e_1 e_0$, and $e_0 e_1 e_0$ up to some power of δ .

To prove the lemma, it remains to prove that those five kinds of normal forms are closed under multiplication by generators e_i . Thanks to Proposition 4.2, we only need consider multiplication from the left. By the conclusion from the above paragraph, the lemma holds. \square

5 The rank of $\text{Im} \phi$

To prove Theorem 1.3, it suffices to prove that those rewritten forms in Proposition 4.3 and 4.6 are different diagrams in $\text{Br}(A_{n-1})$. The problem can be reduced to the counting of orbit sizes.

(I) If $n = 2m - 1$ odd, then

$$\#\phi(W(I_2^{2m-1}))(\{\alpha_{2t}\}_{t=1}^{m-1}) = 2m - 1,$$

(II) If $n = 2m$ even, then

$$\begin{aligned}
\#\phi(W(I_2^{2m}))(\{\alpha_{2t}\}_{t=1}^{m-1}) &= m, \\
\#\phi(W(I_2^{2m}))(\{\alpha_{2t-1}\}_{t=1}^m) &= m, \\
\#\phi(W(I_2^{2m}))(\phi(e_0)\{\alpha_{2t-1}\}_{t=1}^m) &= m,
\end{aligned}$$

and the last two orbits are different.

First we consider $n = 2m - 1 \geq 5$.

By (3.12), when $0 < s < 2m - 1$, we see that $s + 1$ is not occupied in the horizontal strands of $\phi([r_1 r_0 r_1 \cdots]_s^{-1})(\{\alpha_{2t}\}_{t=1}^{m-1})$. Then $\#\phi(W(I_2^{2m-1}))(\{\alpha_{2t}\}_{t=1}^{m-1})$ is at least $2m - 1$. But the subgroup $\langle r_0 \rangle$ stabilizes $\{\alpha_{2t}\}_{t=1}^{m-1}$; therefore by Lagrange's Theorem, (I) holds.

When $n = 2m > 5$, we define

$$\begin{aligned}\alpha &= \sum_{i=1}^{2m-1} \alpha_i, \\ Y_0 &= \{\alpha_{2t}\}_{t=1}^{m-1}, \\ Y_1 &= \{\alpha_{2t-1}\}_{t=1}^m, \\ Y_2 &= \phi(e_0)Y_1 = Y_0 \cup \{\alpha\}.\end{aligned}$$

Now we consider the case when $m = 2m' + 1$ is odd. Here we denote by $h(\gamma)$ the height of $\gamma \in \Phi^+$ which means the sum of the coefficients of simple roots for γ written as the linear combination of simple roots. We find that when $0 \leq s \leq m - 2$,

$$\max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_s^{-1})(Y_0)\} = h(\phi([r_1 r_0 \cdots]_s^{-1})(\alpha_{2m'})) = 2s + 1,$$

$$\max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_{2m'}^{-1})(Y_0)\} = h(\phi([r_1 r_0 \cdots]_{2m'}^{-1})(\alpha_{2m'})) = 4m'.$$

Then it follows that $\#W(I_2^{4m'+2})(Y_0)$ is at least m . At the same time $\langle r_0, [r_1 r_0 \cdots]_{2m-1} \rangle$ stabilizes Y_0 ; therefore it follows from Lagrange's Theorem that $\#W(I_2^{4m'+2})(Y_0)$ is exactly m .

Similarly we see that when $0 \leq s \leq m - 1$

$$\max\{h(\gamma) \mid \gamma \in \phi([r_0 r_1 \cdots]_s^{-1})(Y_1)\} = h(\phi([r_0 r_1 \cdots]_s^{-1})(\alpha_m)) = 2s + 1.$$

Thus $W(I_2^{4m'+2})(Y_1)$ has at least m elements. At the same time $\langle r_1, [r_0 r_1 \cdots]_{2m-1} \rangle$ stabilizes Y_1 , and so by Lagrange's Theorem, $\#W(I_2^{4m'+2})(Y_1) = m$.

When we consider $W(I_2^{4m'+2})(Y_2)$, we need some result from [3, Section 5], we see that Y_2 consists of m' symmetric pairs and 1 symmetric roots, and this numerical information is not changed under $W(I_2^{4m'+2}) \subset W(C_{2m'+1})$ (Weyl group of type $C_{2m'+1}$ in [3, Section 5]). The orbit $W(I_2^{4m'+2})(\{\alpha\})$ has at least m elements, hence using the same argument as the above, we see that $\#W(I_2^{4m'+2})(Y_2)$ is also exactly m .

To prove that the orbits of Y_1 and Y_2 have no intersection, it suffices to verify that Y_2 is not in the orbit of Y_1 . By the above, we see that α only occurs in $Y = \phi([r_0 r_1 \cdots]_{2m'})(Y_1)$ in the orbit of Y_1 under $W(I_2^{2m})$. But $h(\phi([r_0 r_1 \cdots]_{2m'})(\alpha_{m-2})) = 4m' - 2 > 1$, which contradicts the heights of elements in $Y \setminus \{\alpha\}$. With (II) verified, we have proved the Theorem 1.3 for $n \equiv 2 \pmod{4}$, and $n \geq 5$.

At last, consider the case when $n = 2m \geq 5$, and $m = 2m'$. The formula

$\#W(I_2^{4m'})(Y_1) = \#W(I_2^{4m'})(Y_0) = m$ can be proved by the same argument as the above.

From [3, Section 5], we see that Y_1 has m' pairs of symmetric roots and no symmetric root, and Y_2 has $m' - 1$ pairs and 2 symmetric roots α and $\alpha_{2m'}$. Hence the $W(I_2^{2m})$ -orbits of Y_1 and Y_2 have no intersection.

When $0 \leq s < 2m' - 3$, we have

$$\begin{aligned} & \max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_{s+1}^{-1})(Y_2 \setminus \{\alpha_{2m'}, \alpha\})\} \\ & > \max\{h(\gamma) \mid \gamma \in \phi([r_1 r_0 \cdots]_s^{-1})(Y_2 \setminus \{\alpha_{2m'}, \alpha\})\}, \end{aligned}$$

so the orbit of Y_2 has at least $2m' - 2$ elements. Therefore the cardinality of the stabilizer in $W(I_2^n)$ is smaller than $\frac{8m'}{2m'-2}$. If $m' > 3$, $\frac{8m'}{2m'-2} < 6$, but the group $\langle r_0, [r_1 r_0 \cdots]_{4m'-1} \rangle$ stabilizes Y_2 , hence the subgroup will be the full stabilizer. By checking when $m' = 2, 3$, finally, we see that $\#W(I_2^{4m'})(Y_1) = 2m' = m$. With (II) verified, we have proved the main theorem for $n \equiv 0 \pmod{4}$, and $n \geq 5$.

Now we have the following decomposition of $\text{Br}(I_2^n)$ as a $\mathbb{Z}[\delta^{\pm 1}]$ -module.

$$\begin{aligned} \text{Br}(I_2^n) &= \text{Br}(I_2^n)/(e_0) \oplus (e_0), \quad 2 \nmid n, \\ \text{Br}(I_2^n) &= \text{Br}(I_2^n)/(e_0) \oplus (e_0)/(e_0 e_1 e_0) \oplus (e_0 e_1 e_0), \quad 2 \mid n. \end{aligned}$$

Therefore the theorem below about the cellularity can be obtained by an argument similar to [1].

Theorem 5.1. *If R is a field such that the group ring $R[W(I_2^n)]$ is a cellular algebra, then the algebra $\text{Br}(I_2^n) \otimes R$ is a cellularly stratified algebra.*

Remark 5.2. For the hypothesis of the Theorem 5.1, with the method in [5], we conjecture that a sufficient condition for $R[W(I_2^n)]$ is a cellular algebra is that the characteristic of R does not divide n .

Acknowledgment I thank Professor A . M.Cohen for his reading and giving precious suggestion.

References

- [1] C. Bowman, Brauer algebras of type C are cellularly stratified algebras, [arXiv:1102.0438v1](#).
- [2] On algebras which are connected with the semisimple continuous groups, *Annals of mathematics*, Vol 38. No.4 October 1937.
- [3] A. M. Cohen, Shoumin Liu, Shona Yu, Brauer algebra of type C , *Journal of Pure and Applied Algebra*, **216** (2012), 407–426.

- [4] A. M. Cohen, B.Frenk and D.B.Wales, Brauer algebras of simply laced type, Israel Journal of Mathematics, **173** (2009) 335–365.
- [5] M. Geck, Hecke algebras of finite type are cellular, Inventiones Mathematicae, **169** (2007), 501–517.
- [6] S. Liu, Brauer algebras of non-simply laced type, PhD thesis, 2012.
- [7] B. Mühlherr, Coxeter groups in Coxeter groups, pp. 277-287 in Finite Geometry and Combinatorics (Deize 1992). London Math. Soc. Lecture Note Series 191, cambridge university press, Cambridge, 1993.

Shoumin Liu
 University of Amsterdam
 email: s.liu@uva.nl
 liushoumin2003@gmail.com